ISOTROPY OF SOME QUADRATIC FORMS AND ITS APPLICATIONS ON LEVELS AND SUBLEVELS OF ALGEBRAS

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Abstract
In this paper, we give some properties of the levels and sublevels of algebras obtained by the Cayley-Dickson process. We will emphasize how isotropy of some quadratic forms can influence the levels and sublevels on algebras obtained by the Cayley-Dickson process.

1. Introduction
In [18], Pfister showed that if a field has a finite level, this level is a power of 2 and any power of 2 could be realized as the level of a field. In the noncommutative case, the concept of level has many generalizations.
The level of division algebras is defined in the same manner as for the fields.

In this paper, we give some properties of the levels and sublevels of algebras obtained by the Cayley-Dickson process. We will emphasize how isotropy of some quadratic forms can influence the levels and sublevels on algebras obtained by the Cayley-Dickson process.

2. Preliminaries

In this paper, we assume that $K$ is a field and $\text{char } K \neq 2$.

For the basic terminology of quadratic and symmetric bilinear spaces, the reader is referred to [22] or [11]. In this paper, we assume that all the quadratic forms are nondegenerate.

A bilinear space $(V, b)$ represents $\alpha \in K$, if there is an element $x \in V$, $x \neq 0$, with $b(x, x) = \alpha$. The space is called universal, if $(V, b)$ represents all $\alpha \in K$. A vector $x \in V$, $x \neq 0$ is called isotropic if $b(x, x) = 0$, otherwise $x$ is called anisotropic. If the bilinear space $(V, b)$, $V \neq \{0\}$, contains an isotropic vector, then the space is called isotropic. Every isotropic bilinear space $V, V \neq \{0\}$, is universal (see [22], Lemma 4.11, p. 14).

The quadratic form $q : V \to K$ is called anisotropic, if $q(x) = 0$ implies $x = 0$, otherwise $q$ is called isotropic.

A quadratic form $\psi$ is a subform of the form $\phi$, if $\phi \succeq \psi \perp \phi$, for some quadratic form $\phi$. We denote $\psi < \phi$.

Let $\phi$ be an $n$-dimensional quadratic irreducible form over $K$, $n \in N$, $n > 1$, which is not isometric to the hyperbolic plane. We may consider $\phi$ as a homogeneous polynomial of degree 2, $\phi(X) = \phi(X_1, \ldots, X_n) = \sum a_{ij}X_iX_j$, $a_{ij} \in K^*$. The functions field of $\phi$, denoted $K(\phi)$, is the quotient field of the integral domain
\[ K[X_1, \ldots, X_n]/(\varphi(X_1, \ldots, X_n)). \]

Since \((X_1, \ldots, X_n)\) is a non-trivial zero, \(\varphi\) is isotropic over \(K(\varphi)\) (see [22]).

For \(n \in \mathbb{N} - \{0\}\), an \(n\)-fold Pfister form over \(K\) is a quadratic form of the type

\[ <1, a_1 > \otimes \ldots \otimes <1, a_n >, \quad a_1, \ldots, a_n \in K^{*}. \]

A Pfister form is denoted by \(\ll a_1, a_2, \ldots, a_n \gg\). For \(n \in \mathbb{N}, n > 1\), a Pfister form \(\varphi\) can be written as

\[ <1, a_1 > \otimes \ldots \otimes <1, a_n > = <1, a_1, a_2, \ldots, a_n, a_3 a_2 a_3, \ldots, a_2 a_3 \ldots a_n >. \]

If \(\varphi = <1 > \perp \varphi'\), then \(\varphi'\) is called the pure subform of \(\varphi\). A Pfister form is hyperbolic, if and only if it is isotropic. This means that a Pfister form is isotropic, if and only if its pure subform is isotropic (see [22]).

For the field \(L\), it is defined \(L^\infty = L \cup \{x\}\), where \(x + \infty = x\), for \(x \in K, x\infty = \infty\) for \(x \in K^*\), \(\infty\infty = \infty\), \(\frac{1}{\infty} = 0\), and \(\frac{1}{0} = \infty\). An \(L\)-place of the field \(K\) is a map \(\lambda : K \to L^\infty\) with the properties: \(\lambda(x + y) = \lambda(x) + \lambda(y)\), \(\lambda(xy) = \lambda(x)\lambda(y)\), whenever the right sides are defined.

**Theorem** ([8], Theorem 3.3). Let \(F\) be a field of characteristic \(\neq 2\), \(\varphi\) be a quadratic form over \(F\) and \(K, L\) be extensions field of \(F\). If \(\varphi_K\) is isotropic, then there exist an \(F\)-place from \(F(\varphi)\) to \(K\).

An algebra \(A\) over \(K\) is called quadratic, if \(A\) is a unitary algebra and, for all \(x \in A\), there are \(a, b \in K\) such that \(x^2 = ax + b1, a, b \in K\). The subset \(A_0 = \{x \in A - K \mid x^2 \in K1\}\) is a linear subspace of \(A\) and \(A = K \cdot 1 \oplus A_0\).
A composition algebra is an algebra \( A \) with a non-degenerate quadratic form \( q : A \to K \), such that \( q \) is multiplicative, i.e., \( q(xy) = q(x)q(y), \forall x, y \in A \). A unitary composition algebra is called a Hurwitz algebra. Hurwitz algebras have dimensions 1, 2, 4, 8.

Since over fields, the classical Cayley-Dickson process generates all possible Hurwitz algebras, in the following, we briefly present the Cayley-Dickson process and the properties of the obtained algebras.

Let \( A \) be a finite dimensional unitary algebra over a field \( K \), with a scalar involution \( \overline{\cdot} : A \to A, a \to \overline{a} \), i.e., a linear map satisfying the following relations: \( \overline{ab} = \overline{b}\overline{a}, \overline{a} = a, \) and \( a + \overline{a}, a\overline{a} \in K \cdot 1 \) for all \( a, b \in A \). The element \( \overline{a} \) is called the conjugate of the element \( a \), the linear form \( t : A \to K, t(a) = a + \overline{a} \) is called the trace of the element \( a \), and the quadratic form \( n : A \to K, n(a) = a\overline{a} \) is called the norm of the element \( a \). It results that an algebra \( A \) with a scalar involution is quadratic. If the quadratic form \( n \) is anisotropic, then the algebra \( A \) is called anisotropic, otherwise \( A \) is isotropic.

Let \( \gamma \in K \) be a fixed nonzero element. We define the following algebra multiplication on the vector space \( A \oplus A \):

\[
(a_1, a_2)(b_1, b_2) = (a_1b_1 + \gamma \overline{b_2}a_2, \overline{a_2b_1} + b_2a_1).
\] (1.1)

We obtain an algebra structure over \( A \oplus A \). This algebra, denoted by \((A, \gamma)\), is called the algebra obtained from \( A \) by the Cayley-Dickson process. \( A \) is canonically isomorphic with the algebra \( A' = \{(a, 0) \in A \oplus A \mid a \in A\} \), where \( A' \) is a subalgebra of the algebra \((A, \gamma)\). We denote \((1, 0)\) by \( 1 \), where \((1, 0)\) is the identity in \((A, \gamma)\). Taking \( u = (0, 1) \in A \oplus A, u^2 = \gamma \cdot 1 \in K \cdot 1 \), it results that \((A, \gamma) = A \oplus Au\). We have \( \dim(A, \gamma) = 2 \dim A \).
Let $x \in (A, \gamma)$, $x = (a_1, a_2)$. The map

$$- : (A, \gamma) \to (A, \gamma), \quad x \to \bar{x} = (\bar{a}_1, -a_2),$$

is a scalar involution of the algebra $(A, \gamma)$, extending the involution $-$ of the algebra $A$, therefore, the algebra $(A, \gamma)$ is quadratic. For $x \in (A, \gamma)$, $x = (a_1, a_2)$, we denote $t(x) \cdot 1 = x + \bar{x} = t(a_1) \cdot 1 \in K \cdot 1$, $n(x) \cdot 1 = x\bar{x} = (a_1\bar{a}_1 - \gamma a_2\bar{a}_2) \cdot 1 = (n(a_1) - \gamma n(a_2)) \cdot 1 \in K \cdot 1$, and the scalars $t(x) = t(a_1)$, $n(x) = n(a_1) - \gamma n(a_2)$ are called the trace and the norm of the element $x \in (A, \gamma)$, respectively. It follows that

$$x^2 - t(x)x + n(x) = 0, \quad \forall x \in (A, \gamma).$$

If we take $A = K$ and apply this process $t$ times, $t \geq 1$, we obtain an algebra over $K$, $A_t = K\{a_1, \ldots, a_t\}$. By induction, in this algebra, we find a basis $\{1, f_2, \ldots, f_q\}$, $q = 2^t$, satisfying the properties

$$f_i^2 = a_i1, \quad a_i \in K, \quad a_i \neq 0, \quad i = 2, \ldots, q.$$

$$f_if_j = -f_jf_i = \beta_{ij}f_k, \quad \beta_{ij} \in K, \quad \beta_{ij} \neq 0, \quad i \neq j, \quad i, j = 2, \ldots, q, \quad (1.2)$$

$\beta_{ij}$ and $f_k$ being uniquely determined by $f_i$ and $f_j$.

If $x \in A_t$, $x = x_11 + \sum_{i=2}^q x_if_i$, then $\bar{x} = x_11 - \sum_{i=2}^q x_if_i$ and $t(x) = 2x_1$,

$$n(x) = x_1^2 - \sum_{i=2}^q a_ix_i^2.$$  In the above decomposition of $x$, we call $x_1$ the scalar part of $x$ and $x'' = \sum_{i=2}^q x_if_i$ the pure part of $x$. If we compute $x^2 = x_1^2$

$$+ x'^2 + 2x_1x'' = x_1^2 + a_1x_2^2 + a_2x_3^2 - a_1a_2x_4^2 + a_3x_5^2 - \ldots - (-1)^t(\prod_{i=1}^t a_i)x_q^2$$

$$+ 2x_1x'',$$

the scalar part of $x^2$ is represented by the quadratic form
\[ T_C = < 1, \alpha_1, \alpha_2, -\alpha_1 \alpha_2, \alpha_3, ..., (-1)^t(\prod_{i=1}^{t} a_i) > = < 1, \beta_2, ..., \beta_q >, \quad (1.3) \]

and, since \( x^{*2} = a_1 x_2^2 + a_2 x_3^2 - a_1 a_2 x_4^2 + a_3 x_5^2 - ... - (-1)^t(\prod_{i=1}^{t} a_i) x_q^2 \in K, \)

it is represented by the quadratic form \( T_P = T_C |_{A_0}: A_0 \to K, \)

\[ T_P = < \alpha_1, \alpha_2, -\alpha_1 \alpha_2, \alpha_3, ..., (-1)^t(\prod_{i=1}^{t} a_i) > = < \beta_2, ..., \beta_q >. \quad (1.4) \]

The quadratic form \( T_C \) is called the trace form, and \( T_P \) is called the pure trace form of the algebra \( A_t. \) We remark that \( T_C = < 1 > \perp T_P, \) and the norm \( n = n_C = < 1 > \perp -T_P, \) resulting that

\[ n_C = < 1, -\alpha_1, -\alpha_2, \alpha_1 \alpha_2, \alpha_3, ..., (-1)^t+1(\prod_{i=1}^{t} a_i) > = < 1, -\beta_2, ..., -\beta_q >. \]

The norm form \( n_C \) has the form \( n_C = < 1, -\alpha_1 > \otimes ... \otimes < 1, -\alpha_t > \) and it is a Pfister form.

Since the scalar part of any element \( y \in A_t \) is \( \frac{1}{2} t(y), \) it follows that

\[ T_C(x) = \frac{t(x^2)}{2}. \]

**Brown’s construction of division algebras**

In 1967, Brown constructed, for every \( t, \) a division algebra \( A_t \) of dimension \( 2^t \) over the power-series field \( K\{X_1, X_2, ..., X_t\}. \) We will briefly demonstrate this construction, using polynomial rings over \( K \) and their fields of fractions (the rational functions field) instead of power-series fields over \( K \) (as it done by Brown).

First of all, we remark that if an algebra \( A \) is finite-dimensional, then it is a division algebra, if and only if \( A \) does not contain zero divisors (see
For every $t$, we construct a division algebra $A_t$ over a field $F_t$. Let $X_1, X_2, \ldots, X_t$ be $t$ algebraically independent indeterminates over the field $K$ and $F_t = K(X_1, X_2, \ldots, X_t)$ be the rational function field. For $i = 1, \ldots, t$, we construct the algebra $A_i$ over the rational function field $K(X_1, X_2, \ldots, X_i)$ by setting $a_j = X_j$, for $j = 1, 2, \ldots, i$. Let $A_0 = K$. By induction over $i$, assuming that $A_{i-1}$ is a division algebra over the field $F_{i-1} = K(X_1, X_2, \ldots, X_{i-1})$, we may prove that the algebra $A_i$ is a division algebra over the field $F_i = K(X_1, X_2, \ldots, X_i)$.

Let $A_{i-1}^{i-1} = F_i \otimes_{F_{i-1}} A_{i-1}$. For $a_i = X_i$, we apply the Cayley-Dickson process to algebra $A_{i-1}^{i-1}$. The obtained algebra, denoted by $A_i$, is an algebra over the field $F_i$ and has dimension $2^i$.

Let

$$x = a + bv_i, \quad y = c + dv_i,$$

be nonzero elements in $A_i$ such that $xy = 0$, where $v_i^2 = a_i$. Since

$$xy = ac + X_i \bar{d} b + (b\bar{e} + da)\bar{e} = 0,$$

we obtain

$$ac + X_i \bar{d} b = 0, \quad (2.1)$$

and

$$b\bar{e} + da = 0. \quad (2.2)$$

But, the elements $a, b, c, d \in A_{i-1}^{i-1}$ are nonzero elements. Indeed, we have

(i) If $a = 0$ and $b \neq 0$, then $c = d = 0 \Rightarrow y = 0$, false;

(ii) If $b = 0$ and $a \neq 0$, then $d = c = 0 \Rightarrow y = 0$, false;
(iii) If $c = 0$ and $d \neq 0$, then $a = b = 0 \Rightarrow x = 0$, false;

(iv) If $d = 0$ and $c \neq 0$, then $a = b = 0 \Rightarrow x = 0$, false.

This implies that $b \neq 0$, $a \neq 0$, $d \neq 0$, and $c \neq 0$. If $\{1, f_2, \ldots, f_{2^i-1}\}$ is a basis in $A_{i-1}$, then $a = \sum_{j=1}^{2^i-1} g_j (1 \otimes f_j) = \sum_{j=1}^{2^i-1} g_j f_j$, $g_j \in F_i$, $g_j = \frac{g'_j}{g''_j}$, $g'_j, g''_j \in K[X_1, \ldots, X_t], g'_j \neq 0$, $j = 1, 2, \ldots, 2^i-1$, where $K[X_1, \ldots, X_t]$ is the polynomial ring. Let $a_2$ be the least common multiple of $g'_1, \ldots, g'_{2^i-1}$, then we can write $a = \frac{a_1}{a_2}$, where $a_1 \in A_{i-1}^{i-1}$, $a_1 \neq 0$.

Analogously, $b = \frac{b_1}{b_2}$, $c = \frac{c_1}{c_2}$, $d = \frac{d_1}{d_2}$, $b_1, c_1, d_1 \in A_{i-1}^{i-1} - \{0\}$ and $a_2, b_2, c_2, d_2 \in K[X_1, \ldots, X_t] - \{0\}$.

If we replace in the relations (2.1) and (2.2), we obtain

$$a_1 c_1 d_2 b_2 + X_i d_1 b_1 a_2 c_2 = 0, \quad (2.3)$$

and

$$b_1 \bar{c}_1 d_2 a_2 + d_1 a_1 b_2 c_2 = 0. \quad (2.4)$$

If we denote $a_3 = a_1 b_2$, $b_3 = b_1 a_2$, $c_3 = c_1 d_2$, and $d_3 = d_1 c_2$, $a_3, b_3, c_3, d_3 \in A_{i-1}^{i-1} - \{0\}$, the relations (2.3) and (2.4) become

$$a_3 c_3 + X_i d_3 b_3 = 0, \quad (2.5)$$

and

$$b_3 \bar{c}_3 + d_3 a_3 = 0. \quad (2.6)$$

Since the algebra $A_{i-1}^{i-1} = F_i \otimes_{F_{i-1}} A_{i-1}$ is an algebra over $F_{i-1}$ with basis $X^i \otimes f_j$, $i \in \mathbb{N}$ and $j = 1, 2, \ldots, 2^i-1$, we can write $a_3, b_3, c_3, d_3$
under the form $a_3 = \sum_{j \geq m} x_j X_j^i, b_3 = \sum_{j \geq n} y_j X_j^j, c_3 = \sum_{j \geq p} z_j X_j^j,$ and

$d_3 = \sum_{j \geq r} w_j X_j^i,$ where $x_j, y_j, z_j, w_j \in A_{i-1}, x_m, y_n, z_p, w_r \neq 0$. Since $A_{i-1}$ is a division algebra, we have $x_m z_p \neq 0, w_r y_n \neq 0, y_n z_p \neq 0,$ and $w_r x_m \neq 0$. Using the relations (2.5) and (2.6), we have that $2m + p + r = 2n + p + r + 1$, which is false. Therefore, the algebra $A_i$ is a division algebra over the field $F_i = K(X_1, X_2, \ldots, X_i)$ of dimension $2^i$.

3. Main Results

The level of the algebra $A$, denoted by $s(A)$, is the least integer $n$ such that $-1$ is a sum of $n$ squares in $A$. The sublevel of the algebra $A$, denoted by $s(A)$, is the least integer $n$ such that $0$ is a sum of $n + 1$ nonzero squares of elements in $A$. If these numbers do not exist, then the level and sublevel are infinite. Obviously, $s(A) \leq s(A)$.

Let $A$ be a division algebra over a field $K$ obtained by the Cayley-Dickson process, of dimension $q = 2^t$; $T_C, T_P,$ and $n_C$ be its trace, pure trace, and norm forms, respectively.

**Proposition 3.1.** With above notations, we have

(i) If $s(A) \leq n$, then $-1$ is represented by the quadratic form $n \times T_C$.

(ii) $-1$ is a sum of $n$ squares of pure elements in $A$, if and only if the quadratic form $n \times T_P$ represents $-1$.

(iii) For $n \in \mathbb{N} - \{0\}$, if the quadratic form $< 1 > \perp n \times T_P$ is isotropic over $K$, then $s(A) \leq n$.

**Proof.** (i) Let $y \in A, y = x_1 + x_2 f_2 + \ldots + x_q f_q, x_i \in K,$ for all $i \in \{1, 2, \ldots, q\}$. Using the notations given in the Introduction, we get
$y^2 = x_1^2 + \beta_2 x_2^2 + \ldots + \beta_q x_q^2 + 2x_1y^*$, where $y^* = x_2 f_2 + \ldots + x_q f_q$. If $-1$ is a sum of $n$ squares in $A$, then $-1 = y_1^2 + \ldots + y_n^2 = (x_{11}^2 + \beta_2 x_{12}^2 + \ldots + \beta_q x_{1q}^2 + 2x_{11}y_1^*) + \ldots + (x_{n1}^2 + \beta_2 x_{n2}^2 + \ldots + \beta_q x_{nq}^2 + 2x_{n1}y_n^*)$. Then, we have $-1 = \sum_{i=1}^{n} x_{i1}^2 + \beta_2 \sum_{i=1}^{n} x_{i2}^2 + \ldots + \beta_q \sum_{i=1}^{n} x_{iq}^2$ and $\sum_{i=1}^{n} x_{i1}x_{i2} = \sum_{i=1}^{n} x_{i1}x_{i3} = \ldots = \sum_{i=1}^{n} x_{i1}x_{in} = 0$, then $n \times T_C$ represents $-1$.

(ii) With the same notations, if $-1$ is a sum of $n$ squares of pure elements in $A$, then $-1 = y_1^2 + \ldots + y_n^2 = (\beta_2 x_{12}^2 + \ldots + \beta_q x_{1q}^2 + 2x_{11}y_1^*) + \ldots + (\beta_2 x_{n2}^2 + \ldots + \beta_q x_{nq}^2 + 2x_{n1}y_n^*)$. We have $-1 = \beta_2 \sum_{i=1}^{n} x_{i2}^2 + \ldots + \beta_q \sum_{i=1}^{n} x_{iq}^2$. Therefore $n \times T_p$ represents $-1$. Reciprocally, if $n \times T_p$ represents $-1$, then $-1 = \beta_2 \sum_{i=1}^{n} x_{i2}^2 + \ldots + \beta_q \sum_{i=1}^{n} x_{iq}^2$. Let $u_i = x_{i2} f_2 + \ldots + x_{iq} f_q$. It results $t(u_i) = 0$ and $u_i^2 = -n(u_i) = \beta_2 x_{i2}^2 + \ldots + \beta_q x_{iq}^2$, for all $i \in \{1, 2, \ldots, n\}$. We obtain $-1 = u_1^2 + \ldots + u_n^2$.

(iii) Case 1. If $-1 \in K^{+2}$, then $s(A) = 1$.

Case 2. $-1 \notin K^{+2}$. Since the quadratic form $< 1 \perp n \times T_p$ is isotropic, then it is universal. It results that $< 1 \perp n \times T_p$ represent $-1$. Then, we have the elements $\alpha \in K$ and $p_i \in Skew(A)$, $i = 1, \ldots, n$, such that $-1 = \alpha^2 + \beta_2 \sum_{i=1}^{n} p_{i2}^2 + \ldots + \beta_q \sum_{i=1}^{n} p_{iq}^2$, and not all of them are zero.

(i) If $\alpha = 0$, then $-1 = \beta_2 \sum_{i=1}^{n} p_{i2}^2 + \ldots + \beta_q \sum_{i=1}^{n} p_{iq}^2$. It results $-1 = (\beta_2 p_{i2}^2 + \ldots + \beta_q p_{iq}^2) + \ldots + (\beta_2 p_{n2}^2 + \ldots + \beta_q p_{nq}^2)$. Denoting $u_i =$
$p_{12}f_2 + \ldots + p_{iq}f_q$, we have that $t(u_i) = 0$ and $u_i^2 = -n(u_i) = \beta_2 p_{12}^2 + \ldots + \beta_q p_{iq}^2$, for all $i \in \{1, 2, \ldots, n\}$. We obtain $-1 = u_1^2 + \ldots + u_n^2$.

(ii) If $\alpha \neq 0$, then $1 + \alpha^2 \neq 0$ and $0 = 1 + \alpha^2 + \beta_2 \sum_{i=1}^{n} p_{i2}^2 + \ldots + \beta_q \sum_{i=1}^{n} p_{iq}^2$.

Multiplying this relation with $1 + \alpha^2$, it follows that $0 = (1 + \alpha^2)^2 + \beta_2 \sum_{i=1}^{n} r_{i2}^2 + \ldots + \beta_q \sum_{i=1}^{n} r_{iq}^2$.

Therefore $-1 = \beta_2 \sum_{i=1}^{n} r_{i2}^2 + \ldots + \beta_q \sum_{i=1}^{n} r_{iq}^2$, where $r_{ij} = n_j(1 + \alpha)^{-1}$, $j \in \{2, 3, \ldots, q\}$ and we apply case (i). Therefore $s(A) \leq n$.

\[ \Box \]

**Proposition 3.2.** Let $A$ be a division algebra obtained by the Cayley-Dickson process. The following statements are true:

(a) If $n \in \mathbb{N} - \{0\}$, such that $n = 2^k - 1$, for $k > 1$, then $s(A) \leq n$, if and only if $< 1 > T_P$ is isotropic.

(b) If $-1$ is a square in $K$, then $s(A) = s(A) = 1$.

(c) If $-1 \notin K^{*2}$, then $s(A) = 1$, if and only if $T_C$ is isotropic.

**Proof.** (a) From Proposition 3.1, supposing that $s(A) \leq n$, we have

\[ -1 = \sum_{i=1}^{n} p_{i1}^2 + \beta_2 \sum_{i=1}^{n} p_{i2}^2 + \ldots + \beta_q \sum_{i=1}^{n} p_{iq}^2 \text{ such that } \sum_{i=1}^{n} p_{i1} p_{i2} = \sum_{i=1}^{n} p_{i1} p_{i3} = \ldots = \sum_{i=1}^{n} p_{i1} p_{iq} = 0. \]

For the level reasons, it results that $1 + \sum_{i=1}^{n} p_{i1}^2 \neq 0$.

Putting $p_{2^{k_1}} = 1$ and $p_{2^{k_2}} = p_{2^{k_3}} = \ldots = p_{2^{k_q}} = 0$, we have

\[ 0 = \sum_{i=1}^{n+1} p_{i1}^2 + \beta_2 \sum_{i=1}^{n} p_{i2}^2 + \ldots + \beta_q \sum_{i=1}^{n} p_{iq}^2, \quad (3.1) \]
and \( \sum_{i=1}^{n+1} p_{i1} p_{i2} = \sum_{i=1}^{n+1} p_{i1} p_{i3} = \ldots = \sum_{i=1}^{n+1} p_{i1} p_{iq} = 0. \) Multiplying (3.1) by 
\( \sum_{i=1}^{n+1} p_{i1}^2, \) since \( \left( \sum_{i=1}^{n+1} p_{i1}^2 \right)^2 \) is a square and using lemma from [22], p.151, for the products \( \sum_{i=1}^{n+1} p_{i2}^2 \sum_{i=1}^{n+1} p_{i1}^2, \ldots, \sum_{i=1}^{n+1} p_{i2}^2 \sum_{i=1}^{n+1} p_{i1}^2, \) we obtain

\[
0 = \left( \sum_{i=1}^{n+1} p_{i1}^2 \right)^2 + \beta_2 \sum_{i=1}^{n+1} r_{i2}^2 + \ldots + \beta_q \sum_{i=1}^{n+1} r_{iq}^2, \tag{3.2}
\]

where \( r_{i2}, \ldots, r_{iq} \in K, \ n + 1 = 2^k, \ r_{12} = \sum_{i=1}^{n+1} p_{i1} p_{i2} = 0, \ r_{13} = \sum_{i=1}^{n+1} p_{i1} p_{i3} = 0, \ldots, \ r_{1q} = \sum_{i=1}^{n+1} p_{i1} p_{iq} = 0. \) Therefore, in the sums \( \sum_{i=1}^{n+1} r_{i2}, \ldots, \sum_{i=1}^{n+1} r_{iq}, \) we have \( n \) factors. From (3.2), we get that \( <1 > \perp n \times T_p \) is isotropic.

(b) If \( -1 = a^2 \in K \subset A, \) then \( s(A) = s(A) = 1. \)

(c) If \( -1 \not\in K^{*2} \) and \( s(A) = 1, \) then there is an element \( y \in A \setminus K \) such that \( -1 = y^2. \) Hence \( y \in A_0, \) so \( \bar{y} = -y. \) It results that \( (1 + y)^2 = 1 + 2y + y^2 = 2y \) and \( T_C(1 + y) = \frac{1}{2} \left( 1 + y \right)^2 = 1 + (2y + \bar{y}) = y - y = 0. \) Therefore \( T_C \) is isotropic.

Conversely, if \( T_C \) is isotropic, then there is \( y \in A, \ y \neq 0, \) such that \( T_C(y) = 0 = \gamma_1^2 + \beta_2 \gamma_2^2 + \ldots + \beta_q \gamma_q^2. \) If \( \gamma_1 = 0, \) then \( T_C(y) = T_p(y) = 0, \) so \( y = 0, \) which is false. If \( \gamma_1 \neq 0, \) then \( -1 = \left( \frac{\gamma_2}{\gamma_1} f_2 + \ldots + \frac{\gamma_q}{\gamma_1} f_q \right)^2, \)

obtaining \( s(A) = 1. \) \( \square \)
Proposition 3.3. The quadratic form $2^k \times T_C$ is isotropic, if and only if $1 > \perp 2^k \times T_P$ is isotropic.

Proof. Since the form $1 > \perp 2^k \times T_P$ is a subform of the form $2^k \times T_C$, if the form $1 > \perp 2^k \times T_P$ is isotropic, we have that $2^k \times T_C$ is isotropic.

For the converse, supposing that $2^k \times T_C$ is isotropic, then we get

$$\sum_{i=1}^{2^k} p_i^2 + \beta_2 \sum_{i=1}^{2^k} p_{ij}^2 + \ldots + \beta_q \sum_{i=1}^{2^k} p_{iq}^2 = 0,$$

where $p_i, p_{ij} \in K, i = 1, \ldots, 2^k, j = 2, \ldots, q,$ and some of the elements $p_i$ and $p_{ij}$ are nonzero.

If $p_i = 0, \forall i = 1, \ldots, 2^k,$ then $2^k \times T_P$ is isotropic, therefore, $1 > \perp 2^k \times T_P$ is isotropic.

If $\sum_{i=1}^{2^k} p_i^2 \neq 0$, then multiplying relation (3.3) with $\sum_{i=1}^{2^k} p_i^2$ and using lemma from [22], p.151, for the products $\sum_{i=1}^{2^k} p_{ij}^2, \ldots, \sum_{i=1}^{2^k} p_{iq}^2$, we obtain $\left(\sum_{i=1}^{2^k} p_i^2\right)^2 + \beta_2 \sum_{i=1}^{2^k} p_{ij}^2 + \ldots + \beta_q \sum_{i=1}^{2^k} p_{iq}^2 = 0$, then $1 > \perp 2^k \times T_P$ is isotropic.

For the level reason, the relation $\sum_{i=1}^{2^k} p_i^2 = 0$, for some $p_i \neq 0$, does not work. Indeed, supposing that $p_1 \neq 0$, we obtain $-1 = \sum_{i=2}^{2^k} (p_i p_1^{-1})^2$, false. \qed
Proposition 3.4. Let $A$ be an algebra over a field $K$ obtained by the Cayley-Dickson process, of dimension $q = 2^t$, $T_C$ and $T_P$ be its trace and pure trace forms, respectively. If $t \geq 2$ and $2^k \times T_P$ is isotropic over $K$, $k \geq 0$, then $(1 + \lfloor \frac{2}{3}2^k \rfloor) \times T_P$ is isotropic over $K$.

Proof. If $2^k \times T_P$ is isotropic, then $2^k \times -T_P$ is isotropic. Since $2^k \times n_C = 2^k \times (< -1 > \perp -T_P)$ and $n_C$ is a Pfister form, we have $2^k \times n_C$ is a Pfister form. Since $2^k \times -T_P$ is a subform of $2^k \times n_C$, it results that $2^k \times n_C$ is isotropic, then it is hyperbolic. Therefore, $2^k \times n_C \simeq < -1, 1, \ldots, 1, -1, \ldots, -1 >$ (there are $2^{k+t-1}$ of $-1$ and $2^{k+t-1}$ of $1$). Multiplying by $-1$, we have that $2^k \times (< -1 > \perp T_P)$ is hyperbolic, then has a totally isotropic subspace of dimension $2^{k+t-1}$. It results that each subform of the form $2^k \times (< -1 > \perp T_P)$ of dimension greater or equal to $2^{k+t-1}$ is isotropic. Since $(2^t - 1)(1 + \lfloor \frac{2}{3}2^k \rfloor) > (2^t - 1)\lfloor \frac{2}{3}2^k \rfloor > 2^{t-1}2^k = 2^{k+t-1}$, then $(1 + \lfloor \frac{2}{3}2^k \rfloor) \times T_P$ is isotropic over $K$. \qed

Proposition 3.5. Let $A$ be an algebra over a field $K$ obtained by the Cayley-Dickson process, of dimension $q = 2^t$, $T_C$ and $T_P$ be its trace and pure trace forms, respectively. Let $n = 2^k - 1$. If $t \geq 2$ and $k > 1$, then $s_2(A) \leq 2^k - 1$, if and only if $< 1 > \perp (2^k - 1) \times T_P$ is isotropic.

Proof. First, we prove the following result:

Lemma. For $n = 2^k - 1$, $s_2(A) \leq n$, if and only if $< 1 > \perp (n \times T_P)$ is isotropic or $(n + 1) \times T_P$ is isotropic.
**Proof of the Lemma.** Since \( s(A) \leq s(A) \), if \( <1 \perp (n \times T_P) \) is isotropic, then from Proposition 3.2, we have \( s(A) \leq n \). If \( (n + 1) \times T_P \) is isotropic, then there are the elements \( p_{ij} \in K, i = 1, \ldots, q^k, j = 2, \ldots, q, \) some of them are nonzero such that
\[
\beta_2 \sum_{i=1}^{q^k} p_{i2}^2 + \cdots + \beta_q \sum_{i=1}^{q^k} p_{iq}^2 = 0. 
\]
We obtain
\[
0 = (\beta_2 p_{i2}^2 + \cdots + \beta_q p_{iq}^2) + \cdots + (\beta_2 p_{n2}^2 + \cdots + \beta_q p_{nq}^2). 
\]
Denoting \( u_i = p_{i2}f_2 + \cdots + p_{iq}f_q \), we have \( t(u_i) = 0 \) and \( u_i^2 = -n(u_i)^2 = \beta_2 p_{i2}^2 + \cdots + \beta_q p_{iq}^2 \), for all \( i \in \{1, 2, \ldots, n\} \). Therefore \( 0 = u_1^2 + \cdots + u_n^2 \). It results that \( s(A) \leq n \).

Conversely, if \( s(A) \leq n \), then there are the elements \( y_1, \ldots, y_{n+1} \in A \), some of them must be nonzero, such that \( 0 = y_1^2 + \cdots + y_{n+1}^2 \). As in the proof of Proposition 3.1, we obtain
\[
0 = \sum_{i=1}^{n+1} x_{i1}^2 + \beta_2 \sum_{i=1}^{n+1} x_{i2}^2 + \cdots + \beta_q \sum_{i=1}^{n+1} x_{iq}^2. 
\]
and
\[
\sum_{i=1}^{n+1} x_{i1}x_{i2} = \sum_{i=1}^{n+1} x_{i1}x_{i3} = \cdots = \sum_{i=1}^{n+1} x_{i1}x_{iq} = 0. 
\]
If all \( x_{i1} = 0 \), then \( (n + 1) \times T_P \) is isotropic. If \( \sum_{i=1}^{n+1} x_{i1}^2 \neq 0 \), then \( (n + 1) \times T_C \) is isotropic, or multiplying the last relation with \( \sum_{i=1}^{q^k} x_{i1}^2 \) and using lemma from [22], p.151, for the products \( \sum_{i=1}^{q^k} x_{i2}^2 \sum_{i=1}^{q^k} x_{i1}^2, \ldots, \sum_{i=1}^{q^k} x_{iq}^2 \sum_{i=1}^{q^k} x_{i1}^2 \), we obtain that \( <1 \perp (n \times T_P) \) is isotropic. For level reason of the field, the relation
\[
\sum_{i=1}^{n+1} x_{i1}^2 = 0 
\]
for some \( x_{i1} \neq 0 \) is false.
Using the above Lemma, we have that \( s(A) \leq 2^k - 1 \), if and only if \( \langle 1,1 \rangle (n \times T_P) \) is isotropic or \( (n + 1) \times T_P \) is isotropic. In this case, we prove that \( 2^k \times T_P \) is isotropic implies \( \langle 1,1 \rangle (2^k - 1) \times T_P \) is isotropic.

If \( 2^k \times T_P \) is isotropic over \( K \), then \( (\{ \frac{2}{3}, 2^k \}) \times T_P \) is isotropic over \( K \).

If \( k \geq 2 \), then \( (1 + \{ \frac{2}{3}, 2^k \}) \leq 2^k - 1 \) and we have that \( (1 + \{ \frac{2}{3}, 2^k \}) \times T_P \) is an isotropic subform of the form \( \langle 1,1 \rangle (2^k - 1) \times T_P \).

\[ \square \]

**Remark 3.6.** Using the above notations, if the algebra \( A \) is an algebra obtained by the Cayley-Dickson process, of dimension greater than 2 and if \( n_C \) is isotropic, then \( s(A) = s(A) = 1 \). Indeed, if \( -1 \) is a square in \( K \), the statement results from above. If \( -1 \notin K^{\ast,2} \), since \( n_C = \langle 1,1 \rangle \) and \( n_C \) is a Pfister form, we obtain that \( -T_P \) is isotropic, therefore \( T_C \) is isotropic and, from above proposition, we have that \( s(A) = s(A) = 1 \).

**Proposition 3.7.** Let \( A \) be an algebra over a field \( K \) obtained by the Cayley-Dickson process, of dimension \( q = 2^t \), \( T_C \) and \( T_P \) be its trace and pure trace forms, respectively. If \( k \geq t \), then \( s(A) \leq 2^k \), if and only if the form \( (2^k + 1) \times \langle 1,1 \rangle (2^k - 1) \times T_P \) is isotropic.

**Proof.** First, we prove the following result:

**Lemma.** \( s(A) \leq 2^k = n \), if and only if \( (2^k + 1) \times \langle 1,1 \rangle (2^k - 1) \times T_P \) is isotropic or \( \langle 1,1 \rangle 2^k \times T_P \) is isotropic.

**Proof of the Lemma.** If \( s(A) \leq 2^k \), therefore \( -1 = y_1^2 + \ldots + y_n^2 \), where \( y_i \in A \). Using the above notations, we obtain

\[
0 = 1 + \sum_{i=1}^{n} x_{i1}^2 + \beta_2 \sum_{i=1}^{n} x_{i2}^2 + \ldots + \beta_q \sum_{i=1}^{n} x_{iq}^2, \tag{\ast}
\]
and \( \sum_{i=1}^{n} x_{i1} x_{i2} = \sum_{i=1}^{n} x_{i1} x_{i3} = \ldots = \sum_{i=1}^{n} x_{i1} x_{iq} = 0 \). If \( \sum_{i=1}^{n} x_{i1}^2 = 0 \), then

\(< 1 >_{\perp} 2^k \times T_P \). If \( 1 + \sum_{i=1}^{n} x_{i1}^2 = 0 \), we have \( 2^k \times T_P \) is isotropic, therefore

\(< 1 >_{\perp} 2^k \times T_P \) is isotropic. If \( a = \sum_{i=1}^{n} x_{i1}^2 \neq 0 \), multiplying relation (*) with

\[ \sum_{i=1}^{n} x_{i1}^2 \] and using lemma from [22], p.151, for the products \( \sum_{i=1}^{n} x_{i2}^2 \sum_{i=1}^{n} x_{i1}^2 \),

\[ \ldots, \sum_{i=1}^{n} x_{i1}^2 \sum_{i=1}^{n} x_{i1}^2, \]

we obtain \( \sum_{i=1}^{n} x_{i1}^2 + a^2 + \beta_2 \sum_{i=1}^{n} r_{i2}^2 + \ldots + \beta_q \sum_{i=1}^{n} r_{iq}^2 = 0 \),

where \( r_{i2}, \ldots, r_{iq} \in K, r_{i2} = \sum_{i=1}^{n} x_{i1} x_{i2} = 0, r_{i3} = \sum_{i=1}^{n} x_{i1} x_{i3} = 0, \ldots, r_{iq} \)

\[ = \sum_{i=1}^{n} p_{i1} p_{iq} = 0. \] Therefore, in the sums \( \sum_{i=1}^{n-1} r_{i2}^2, \ldots, \sum_{i=1}^{n-1} r_{iq}^2 \), we have \( n - 1 \) factors. It results that \( (2^k + 1) \times < 1 >_{\perp} (2^k - 1) \times T_P \) is isotropic.

Conversely, if \( < 1 >_{\perp} 2^k \times T_P \) is isotropic, from Proposition 3.1 (iii), we have \( s(A) \leq 2^k \). If \( (2^k + 1) \times < 1 >_{\perp} (2^k - 1) \times T_P \) is isotropic, then there is a nonzero element \( a \in D_K(2^k \times < 1 >) \) such that \(-a \in D_K(< 1 >_{\perp} (2^k - 1) \times T_P \)). It results that \(-a = b_2^2 + \beta_2 B_2 \ldots + \beta_q B_q \), where \( b \in K, B_2, \ldots, B_q \in D_K((2^k - 1) \times < 1 >) \cup \{0\} \), so that \(-1 = \frac{1}{a^2} (b^2 a + \beta_2 B_2 a + \ldots + \beta_q B_q a) \). Supposing that \( a = \sum_{i=1}^{n} x_{i1}^2 \), there are the elements \( y_{i2} \in K \) such that \( \sum_{i=1}^{n} y_{i2}^2 = aB \) and \( \sum_{i=1}^{n} x_i y_{i2} = 0 \). Indeed, if \( B_2 = 0 \), we put \( y_{i2} = 0, i \in \{1, 2, \ldots, n\} \). If \( B_2 \neq 0 \), we have \( \sum_{i=1}^{n} x_i y_{i2} = 0 \).
\[ \alpha < 1, B_2 > . \text{ Since } B_2 \in D_K((2^k - 1) \times < 1 >) \text{ and } \alpha \cdot 2^k \times < 1 > \simeq 2^k \times < 1 >, \]
it results that \[ \alpha < 1, B_2 > \text{ is a subform of } 2^k \times < 1 >, \]
then the elements \( y_{i2} \) exist. In the same way, we get the elements \( y_{ij} \in K, j \in \{3, \ldots, q\} \)
such that \[ aB_j = \sum_{i=1}^{n} y_{ij}^2 \text{ and } \sum_{i=1}^{n} x_i y_{ij} = 0, \]
obtaining that \[ \sum_{i=1}^{n} \frac{1}{a^2} (ax_i + y_{i2}f_2 + \ldots + y_{iq}f_q) = \frac{1}{a^2} (b^2a + \beta_2B_2a + \ldots + \beta_qB_qa) = -1, \]
then \( s(A) \leq n. \]

Using the above Lemma, we have \( s(A) \leq 2^k \), if and only if \( (2^k + 1) \times < 1 > \perp (2^k - 1) \times T_P \) or \( < 1 > \perp 2^k \times T_P \) is isotropic. If \( k \geq t \), we prove that \( < 1 > \perp 2^k \times T_P \) isotropic implies \( (2^k + 1) \times < 1 > \perp (2^k - 1) \times T_P \) isotropic. Indeed, \( < 1 > \perp 2^k \times T_P \) is isotropic, if and only if \( 2^k \times T_C \) is isotropic, from Proposition 3.3. From [3], Proposition 1.4, it results that the Witt index (i.e., the dimension of maximal totally isotropic subspace) of \( 2^k \times T_C \) is greater or equal with \( 2^k \). Therefore, \( 2^k \times T_C \) has a totally isotropic subspace of dimension \( \geq 2^k \). The form \( 2^k \times < 1 > \perp (2^k - 1) \times T_P \) is a subform of the forms \( 2^k \times T_C \) and \( (2^k + 1) \times < 1 > \perp (2^k - 1) \times T_P \) and has dimension \( (2^t - 1)(2^k - 1) + 2^k \). Since \( (2^t - 1)(2^k - 1) + 2^k = 2^{k+t} - 2^t + 1 > 2^k \), for \( k \geq t \), we have that \( 2^k \times < 1 > \perp (2^k - 1) \times T_P \) is isotropic, then \( (2^k + 1) \times < 1 > \perp (2^k - 1) \times T_P \) is isotropic. \( \square \)

**Proposition 3.8.** Let \( K \) be a field.

(i) If \( k \geq 2 \), then \( s(A) \leq 2^k - 1 \), if and only if \( s(A) \leq 2^k - 1 \).

(ii) If \( s(A) = n \) and \( k \geq 2 \) such that \( 2^{k-1} \leq n < 2^k \), then \( s(A) \leq 2^k - 1 \).

(iii) If \( s(A) = 1 \), then \( s(A) \leq 2 \).
Proof. (i) For $k \geq 2$, then $s(A) \leq 2^k - 1$, if and only if $<1>_{\perp}$ $(2^k - 1) \times T_P$ is isotropic. This is equivalent with $s(A) \leq 2^k - 1$.

(ii) If $n < 2^k$, $k \geq 2$, it results $n \leq 2^k - 1$, and we apply (i).

(iii) We have that $s(A) = 1$, if and only if $<1>_{\perp} T_P = T_C$ is isotropic or $2 \times T_P$ is isotropic. If $2 \times T_P$ is isotropic, then it is universal and represents $-1$. Therefore $s(A) \leq 2$. If $T_C$ is isotropic, then $T_P$ is isotropic, then it is universal and represents $-1$. We obtain $s(A) = 1$. □

Proposition 3.9. With the above notations, we have

(i) For $k \geq 2$, if $s(A) = 2^k - 1$, then $s(A) = 2^k - 1$.

(ii) For $k \geq 2$, if $s(A) = 2^k$, then $s(A) = 2^k$.

(iii) For $k \geq 1$, if $s(A) = 2^k + 1$, then $s(A) = 2^k + 1$ or $s(A) = 2^k$.

Proof. (i) From Proposition 3.8, if $s(A) = 2^k - 1$, then $s(A) \leq 2^k - 1$. Since $s(A) \leq s(A)$, therefore $s(A) = 2^k - 1$.

(ii) If $s(A) \leq 2^k - 1$, we have $s(A) \leq 2^k - 1$, false.

(iii) For $k \geq 1$, if $s(A) = 2^k + 1$, since $s(A) \leq s(A)$, we obtain that $s(A) \leq 2^k + 1$. If $s(A) \leq 2^k - 1$, then $s(A) \leq 2^k - 1$, false. □

Some of the above results are proved by O’Shea in [16] and in [17] for the quaternions and octonions.

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References


